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AUTOMORPHISMS OF DISTANCE-REGULAR GRAPH WITH INTERSECTION ARRAY $\{25, 16, 1; 1, 8, 25\}$ [1]

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Abstract: Makhnev and Samoilenko have found parameters of strongly regular graphs with no more than 1000 vertices, which may be neighborhoods of vertices in antipodal distance-regular graph of diameter 3 and with $\lambda = \mu$. They proposed the program of investigation vertex-symmetric antipodal distance-regular graphs of diameter 3 with $\lambda = \mu$, in which neighborhoods of vertices are strongly regular. In this paper we consider neighborhoods of vertices with parameters $(25, 8, 3, 2)$.

Key words: Strongly regular graph, Distance-regular graph.

Introduction

We consider undirected graphs without loops and multiple edges. Given a vertex a in a graph Γ, we denote by $\Gamma_i(a)$ the subgraph induced by Γ on the set of all vertices, that are at the distance i from a. The subgraph $[a] = \Gamma_1(a)$ is called the *neighborhood of the vertex* a. Let $\Gamma(a) = \Gamma_1(a)$, $a^\perp = \{a\} \cup \Gamma(a)$. If graph Γ is fixed, then instead of $\Gamma(a)$ we write $[a]$. For the set of vertices X of graph Γ through $X^\perp$ denote $\cap_{x\in X} x^\perp$.

Let Γ be an antipodal distance-regular graph of diameter 3 and $\lambda = \mu$, in which neighborhoods of vertices are strongly-regular graphs. Then Γ has intersection array $\{k, \mu(r-1), 1; 1, \mu, k\}$, and spectrum $k^1, \sqrt{k}^f, -1^k, -\sqrt{k}^f$, where $f = (k+1)(r-1)/2$. In the case $r = 2$ we obtain Taylor's graph, in which $k' = 2\mu'$. Conversely, for any strongly regular graph with parameters $(v', 2\mu', \lambda', \mu')$ there exists a Taylor's graph, in which neighborhoods of vertices are strongly regular with relevant parameters.

In [1]there were chosen strongly-regular graphs with no more than 1000 vertices, which may be neighborhoods of vertices of antipodal distance-regular graph of diameter 3 and $\lambda = \mu$. There is provided a research program of the study of vertex-symmetric antipodal distance-regular graphs of diameter 3 with $\lambda = \mu$, in which neighborhoods of vertices are strongly regular with parameters from Proposition 1.

Proposition 1. *Let Δ be a strongly-regular graph with parameters (v, k, λ, μ). If $(r-1)k = v-k-1$, $v \le 1000$ and number $(v+1)(r-1)$ is even, then either $r = 2$, or parameters (v, k, λ, μ, r) belong to the following list:*

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(1) $(16,5,0,2,3)$, $(25,8,3,2,3)$, $(49,12,5,2,4)$, $(64,21,8,6,3)$, $(81,16,7,2,5)$, $(81,20,1,6,4)$, $(85,14,3,2,6)$, $(99,14,1,2,7)$, $(100,33,8,12,3)$, $(121,20,9,2,6)$, $(121,30,11,6,4)$, $(121,40,15,12,3)$, $(126,25,8,4,5)$, $(133,44,15,14,3)$, $(169,24,11,2,7)$, $(169,42,5,12,4)$, $(169,56,15,20,3)$, $(176,25,0,4,7)$, $(196,39,14,6,5)$, $(196,65,24,20,3)$;

(2) $(225,28,13,2,8)$, $(225,56,19,12,4)$, $(243,22,1,2,11)$, $(256,51,2,12,5)$, $(256,85,24,30,3)$, $(261,52,11,10,5)$, $(288,41,4,6,7)$, $(289,32,15,2,9)$, $(289,48,17,6,6)$, $(289,72,11,20,4)$, $(289,96,35,30,3)$, $(305,76,27,16,4)$, $(325,54,3,10,6)$, $(351,50,13,6,7)$, $(351,70,13,14,5)$, $(352,39,6,4,9)$, $(361,36,17,2,10)$, $(361,72,23,12,5)$, $(361,90,29,20,4)$, $(361,120,35,42,3)$;

(3) $(400,57,20,6,7)$, $(400,133,48,42,3)$, $(441,40,19,2,11)$, $(441,88,7,20,5)$, $(441,110,19,30,4)$, $(484,161,48,56,3)$, $(495,38,1,3,13)$, $(505,84,3,16,6)$, $(507,46,5,4,11)$, $(512,73,12,10,7)$, $(529,44,21,2,12)$, $(529,66,23,6,8)$, $(529,88,27,12,6)$, $(529,132,41,30,4)$, $(529,176,63,56,3)$, $(540,49,8,4,11)$, $(576,115,18,24,5)$;

(4) $(625,48,23,2,13)$, $(625,156,29,42,4)$, $(625,208,63,72,3)$, $(640,71,6,8,9)$, $(649,72,15,7,9)$, $(649,216,63,76,3)$, $(676,75,26,6,9)$, $(676,135,14,30,5)$, $(704,37,0,2,19)$, $(729,52,25,2,14)$, $(729,104,31,12,7)$, $(729,182,55,42,4)$, $(736,105,20,14,7)$, $(768,59,10,4,13)$, $(784,261,80,90,3)$;

(5) $(837,76,15,6,11)$, $(841,56,27,2,15)$, $(841,84,29,6,10)$, $(841,140,39,20,6)$, $(841,168,47,30,5)$, $(841,210,41,56,4)$, $(841,280,99,90,3)$, $(847,94,21,9,9)$, $(848,121,24,16,7)$, $(901,60,3,4,15)$, $(961,60,29,2,16)$, $(961,120,35,12,8)$, $(961,160,9,30,6)$, $(961,192,23,42,5)$, $(961,240,71,56,4)$, $(961,320,99,100,3)$, $(1000,111,14,12,9)$.

Graphs with local subgraphs having parameters $(64,21,8,6)$, $(81,16,7,2)$, $(85,14,3,2)$ and $(99,14,1,2)$ were investigated in [2], [3], [4] and [5]. In this article we investigate parameters $(25,8,3,2,3)$, i.e. this graph is locally 5×5-grid. In [6] it is proved that distance-regular locally 5×5-grid of diameter more then 2 is either isomorphic to the Johnson's graph $J(10,5)$ or has an intersection array $\{25,16,1;1,8,25\}$.

Theorem 1. *Let Γ be a distance-regular graph with intersection array $\{25,16,1;1,8,25\}$, $G=\mathrm{Aut}(\Gamma)$, g is an element of prime order p in G and $\Omega=\mathrm{Fix}(g)$ contains exactly s vertices in t antipodal classes. Then $\pi(G)\subseteq\{2,3,5,13\}$ and one of the following assertions holds:*

(1) *Ω is empty graph and $p\in\{2,3,13\}$;*

(2) *$p=5$, $t=1$, $\alpha_3(g)=0$, $\alpha_1(g)=50l+25$ and $\alpha_2(g)=50-50l$;*

(3) *$p=3$, $s=3$, $t=2,5,8$, $\alpha_3(g)=0$, $\alpha_1(g)=30l+16-11t$ and $\alpha_2(g)=62-30l+8t$;*

(4) *$p=2$, and either $s=1$, Ω is t-clique, $t=2,4,6$, $\alpha_3(g)=2t$, $\alpha_1(g)=20l-t+6$ and $\alpha_2(g)=72-20l-2t$, or $s=3$, $t\le 8$, t is even, $\alpha_3(g)=0$, $\alpha_1(g)=20l-11t+6$ and $\alpha_2(g)=72-20l+8t$.*

Corollary 1. *Let Γ be a distance-regular graph with intersection array $\{25,16,1;1,8,25\}$ and a group $G=\mathrm{Aut}(\Gamma)$ acts transitively on the set of vertices of Γ. Then one of the following assertions holds:*

(1) *Γ is a Cayley graph, G is the a Frobenius group with the kernel of order 13 and with the complement of order 6;*

(2) *Γ is a arc-transitive Maton's graph and the socle of G is isomorphic to $L_2(25)$;*

(3) *G is an extension of a group Q of order 2^{12} by the group $T=L_3(3)$, $|Q:Q_{\{F\}}|=2$, $T_{\{F\}}$ is an extension of group E_9 by $SL_2(3)$, T acts irreducibly on Q and for an element f of order 13 in G we have $C_Q(f)=1$.*

1. Proof of the Theorem

Note that there is Delsarte boundary (proposition 4.4.6 from [7]) of maximum order of clique in distance-regular graph with intersection array $\{25, 16, 1; 1, 8, 25\}$ and spectrum $25^1, 5^{26}, -1^{25}, -5^{26}$ no more than $1 - k/\theta_d = 1 + 25/5 = 6$. If C is 6-clique in Γ, then each vertex not in C is adjacent to 0 or to $b_1/(\theta_d + 1) + 1 - k/\theta_d = 2$ vertices in C.

Lemma 1. *Let Γ be a distance-regular graph with intersection array $\{25, 16, 1; 1, 8, 25\}$, $G = \mathrm{Aut}(\Gamma)$ and $g \in G$. If ψ is the monomial representation of a group G in $GL(78, \mathbf{C})$, χ_1 is the character of the representation ψ on subspace of eigenvectors of dimension 26, corresponding to the eigenvalue 5, χ_2 is the character of the representation ψ on subspace of dimension 25, then $\chi_1(g) = (10\alpha_0(g) + 2\alpha_1(g) - \alpha_2(g) - 5\alpha_3(g))/30$, $\chi_2(g) = (\alpha_0(g) + \alpha_3(g))/3 - 1$. If $|g| = p$ is prime, then $\chi_1(g) - 26$ and $\chi_2(g) - 25$ are divided by p.*

P r o o f. We have

$$Q = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 26 & 26/5 & -13/5 & -13 \\ 25 & -1 & -1 & 25 \\ 26 & -26/5 & 13/5 & -13 \end{pmatrix}.$$

Therefore $\chi_1(g) = (10\alpha_0(g) + 2\alpha_1(g) - \alpha_2(g) - 5\alpha_3(g))/30$. Substituting $\alpha_2(g) = 78 - \alpha_0(g) - \alpha_1(g) - \alpha_3(g)$, we obtain $\chi_1(g) = (11\alpha_0(g) + 3\alpha_1(g) - 4\alpha_3(g))/30 - 13/5$.

Similarly, $\chi_2(g) = (25\alpha_0(g) - \alpha_1(g) - \alpha_2(g) + 25\alpha_3(g))/78$. Substituting $\alpha_1(g) + \alpha_2(g) = 78 - \alpha_0(g) - \alpha_3(g)$, we obtain $\chi_2(g) = (\alpha_0(g) + \alpha_3(g))/3 - 1$.

The remaining assertions follow from Lemma 1 in [8]. The proof is complete. □

Let further in the paper Γ be a distance-regular graph with intersection array $\{25, 16, 1; 1, 8, 25\}$, $G = \mathrm{Aut}(\Gamma)$, g is an element of prime order p in G and $\Omega = \mathrm{Fix}(g)$.

Lemma 2. *If Ω is an empty graph, then either $p = 13$, $\alpha_1(g) = 26$ and $\alpha_2(g) = 52$, or $p = 3$, $\alpha_3(g) = 9s+6$, $s < 8$, $\alpha_1(g) = 54+12s-30l$ and $\alpha_2(g) = 18-21s+30l$, $l \leq 5$, or $p = 2$, $\alpha_3(g) = 0$, $\alpha_1(g) = 20l + 6$ and $\alpha_2(g) = 72 - 20l$, $l \leq 3$.*

P r o o f. Let Ω be an empty graph and $\alpha_i(g) = pw_i$ for $i > 0$. Since $v = 78$, we have $p \in \{2, 3, 13\}$.

Let $p = 13$. Then $\alpha_3(g) = 0$, $\alpha_1(g) + \alpha_2(g) = 78$ and $\chi_1(g) = (2\alpha_1(g) - \alpha_2(g))/30 = 13(w_1 - 2)/10$. This implies $\alpha_1(g) = 26$ and $\alpha_2(g) = 52$.

Let $p = 3$. Then $\chi_2(g) - 25 = \alpha_3(g)/3 - 26$ is divided by 3, $\alpha_3(g) = 9s+6$, $s \leq 8$ and $\alpha_2(g) = 72 - 9s - \alpha_1(g)$. Furthermore, the number $\chi_1(g) = (2\alpha_1(g) - \alpha_2(g) - 45s - 30)/30 = (3w_1 - 12s - 34)/10$ is congruent to 2 modulo 3. This implies $\alpha_1(g) = 54 + 12s - 30l$ and $\alpha_2(g) = 18 - 21s + 30l$, $l \leq 5$. In case $s = 8$ we have $\alpha_3(g) = 78$ and $\langle g \rangle$ acts regularly on each antipodal class. By lemma 4 in [9] 3 must divide $k + 1 = 26$, we have a contradiction.

Let $p = 2$. Then $\alpha_3(g) = 0$, $\alpha_1(g) + \alpha_2(g) = 78$, the number $\chi_1(g) = (\alpha_1(g) - 26)/10$ is even, $\alpha_1(g) = 20l + 6$ and $\alpha_2(g) = 72 - 20l$, $l \leq 3$. □

In Lemmas 3–6 it is assumed that there are t antipodal classes intersecting the Ω on s vertices. Then p divides $26 - t$ and $3 - s$. Let F be an antipodal class, containing the vertex $a \in \Omega$, $F \cap \Omega = \{a, a_2, ..., a_s\}$, $b \in \Omega(a)$. By $F(x)$ we denote an antipodal class containing vertex x.

Lemma 3. *The following assertions hold:*
(1) *if $t=1$, then $p=5$, $\alpha_3(g)=0$, $\alpha_1(g)=50l+25$ and $\alpha_2(g)=50-50l$;*
(2) *if p more than 3, then $p=5$ and $t=1$;*
(3) *if $s=1$, then $p=2$, $t=2,4,6$, $\alpha_3(g)=2t$, $\alpha_1(g)=20l-t+6$ and $\alpha_2(g)=72-20l-2t$.*

P r o o f. If $s=3$, then each vertex from $\Gamma-\Omega$ is adjacent to t vertices in Ω, so $t\le 8$.

Let $t=1$. As p divides $26-t$, then $p=5$, $s=3$, $\alpha_2(g)=75-\alpha_1(g)$, the number $\chi_1(g)=(\alpha_1(g)-15)/10$ is congruent to 1 modulo 5. This implies $\alpha_1(g)=50l+25$.

Let $p>3$, $\alpha_1(g)=pw_1$. Then $s=3$, $|\Omega|=3t$, Ω is a regular graph by degree $t-1$ and p divides $26-t$.

If $p>7$, then Ω is a distance-regular graph with intersection array $\{t-1,16,1;1,8,t-1\}$, we come to a contradiction.

Let $p=7$. As p divides $26-t$, then $t=5$, the subgraph $\Omega(b)$ contains 2 vertices in $a^\perp$ and a vertex from $[a_2]$ and from $[a_3]$, so Ω is a distance-regular graph with intersection array $\{4,1,1;1,1,4\}$, it is a contradiction with the fact that $r=3$.

Let $p=5$. As p divides $26-t$, then $t=1,6$. If $t=6$, then the subgraph $\Omega(b)$ contains a vertex in $a^\perp$, 3 vertices from $[a_2]$ and 3 vertices from $[a_3]$, we come to a contradiction.

Let $s=1$. Then $p=2$, $t\le 6$, $\alpha_3(g)=2t$, $\alpha_2(g)=78-\alpha_1(g)-3t$, and $\chi_1(g)=(\alpha_1(g)+t-26)/10$ is even. This implies that $\alpha_1(g)=20l-t+6$. □

Lemma 4. *If $p=3$, then $s=3$, $t=2,5,8$, $\alpha_3(g)=0$, $\alpha_1(g)=30l+16-11t$ and $\alpha_2(g)=62-30l+8t$.*

P r o o f. Let $p=3$. Then $s=3$, $t=2,5,8$, $\alpha_2(g)=78-\alpha_1(g)-3t$, and the number $\chi_1(g)=(11t+\alpha_1(g)-26)/10$ is congruent to 2 modulo 3. This implies that $\alpha_1(g)=30l+16-11t$. In the case $t=2$ graph Ω is a union of 3 isolated edges. □

Lemma 5. *If $p=2$, $s=3$, then t is even, $t\le 8$, $\alpha_3(g)=0$, $\alpha_1(g)=20l-11t+6$ and $\alpha_2(g)=72-20l+8t$.*

P r o o f. Let $p=2$, $s=3$. Then t is even, $t\le 8$, $\alpha_3(g)=0$, $\alpha_2(g)=78-3t-\alpha_1(g)$.
The number $\chi_1(g)=(11t+\alpha_1(g)-26)/10$ is even, so $\alpha_1(g)=20l-11t+6$. □

Lemmas 2–5 imply the proof of the Theorem.

2. Proof of Corollary

Let the group G acts transitively on the set of vertices of the graph Γ. Then for a vertex $a\in\Gamma$ subgroup $H=G_a$ has index 78 in G. By Theorem we have $\{2,3,13\}\subseteq\pi(G)\subseteq\{2,3,5,13\}$.

Lemma 6. *Let f be an element of order* 13 *in G. Then* $\mathrm{Fix}(f)$ *is an empty graph, $\alpha_1(f)=26$ and the following assertions hold:*
(1) *if g is an element of prime order $p\ne 13$ in $C_G(f)$, then $p=2$, Ω is an empty graph, $\alpha_1(g)=26$ and $|C_G(f)|$ is not divided by 4;*
(2) *either $|G|=78$ or $F(G)=O_2(G)$;*
(3) *if G is nonsolvable group, then the socle $\bar T$ of the group $\bar G=G/F(G)$ is isomorphic to $L_2(25)$, $L_3(3)$, $U_3(4)$, $L_4(3)$ or ${}^2F_4(2)'$.*

P r o o f. By Lemma 2 Fix(f) is an empty graph and $\alpha_1(f) = 26$.

Suppose that g is an element of prime order $p \neq 13$ in $C_G(f)$. As f acts without fixed points on Ω then by Theorem Ω is an empty graph, $p = 2$ and $\alpha_1(g) = 20l + 6$ divided by 13. This implies that $\alpha_1(g) = 26$ and $|C_G(f)|$ is not divided by 4.

Let $Q = O_p(G) \neq 1$. If $p = 13$, then $|G|$ divides $26 \cdot 12$. In this case $C_G(f) = \langle f \rangle$, otherwise for an involution g of $C_G(f)$ we obtain a contradiction with the action of element of order 3 of G on $\{u \mid d(u, u^g) = 1\}$. Let the involution g inverts f, h is an element of order 3 in $C_G(g)$. From action h on $\{u \mid d(u, u^g) = 1\}$ it follows that $\alpha_1(g) = 20l + 6$ is divided by 3. In each case $\alpha_1(g)$ is not divided by 4 and $|G| = 78$.

If $p = 3$, then Q fixes some antipodal class. This implies that Q fixes each antipodal class. By Lemma 3 in [9] G does not contain subgroups of order 3, which are regular on each antipodal class, we come to a contradiction. So, if $|G| \neq 78$ we have $F(G) = O_2(G)$.

Let $\bar{T}$ be the socle of the group $\bar{G} = G/F(G)$. Note that 13 divides $|\bar{T}|$ and by Theorem 1 in [10] group $\bar{T}$ is isomorphic to $L_2(25)$, $L_3(3)$, $U_3(4)$, $L_4(3)$, ${}^2F_4(2)'$. □

Let us to prove the Corollary. As $\bar{T}$ contains a subgroup of index dividing 26, then the group $\bar{T}$ is isomorphic to $L_2(25)$ (and $\bar{T}_{\{F\}}$ is the extension of a group of order 25 by group of order 12) or $L_3(3)$ (and $\bar{T}_{\{F\}}$ is the extension of a group of order 9 by $SL(2,3)$).

In the first case $F(G)$ fixes each antipodal class and $F(G) = 1$. This implies that Γ is the arc-transitiv Maton's graph.

In the second case for $Q = F(G)$ we have $|Q : Q_{\{F\}}| = 2$ and $\bar{T}$ acts irreducibly on Q. Further, for the element f of order 13 of G by Lemma 6 the number $|C_Q(f)|$ divides 2. As Q is either 12-dimensional module over F_2, or 16-dimensional module over F_{16}, or 26-dimensional module over F_2, then $|Q| = 2^{12}$ and $C_Q(f) = 1$. The Corollary is proved.

3. Conclusion

We found possible automorphisms of a distance regular graph with intersection array $\{25, 16, 1; 1, 8, 25\}$. This completes the research program of vertex-symmetric antipodal distance-regular graphs of diameter 3 with $\lambda = \mu$, in which neighborhoods of vertices are strongly regular with parameters from Proposition 1.

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